

QUASI-BOOLEAN POWER OF ALGEBRAS AND  
IDEMPOTENT ALGEBRAS

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In this paper we provide a necessity condition for embedding of the binary algebra into the quasi-boolean power of a rectangular algebra. It is also proved that every idempotent and hyperassociative algebra via the weak bihomomorphism maps in an idempotent and commutative algebra.

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**Preliminaries.** The concept of the boolean power of the algebra plays an important role in the general theory of algebraic systems [1]. A close generalization of this concept is the quasi-boolean power of algebra.

**Definition 1.** Let  $L(+, \cdot)$  be a complete lattice.  $\lambda = \{l_i \in L \mid i \in I\}$  subset of  $L$  is called orthogonal system if  $l_i \cdot l_j = 0$ , where  $i \neq j$ .

**Definition 2.** Let  $L(+, \cdot)$  be a complete lattice.  $\lambda = \{l_i \in L \mid i \in I\}$  orthogonal system is called independent if  $\left(\sum_{j \in J} l_j\right) \cdot \left(\sum_{k \in K} l_k\right) = 0$ , where  $J \cup K = I$ ,  $J \cap K = \emptyset$ .

**Definition 3.** The complete, complemented lattice is called quasi-boolean lattice, if its each orthogonal system is independent.

All complete boolean lattices are quasi-boolean lattices.

In the future we will consider algebras with binary operations only. Let  $L$  be a quasi-boolean lattice and  $S = (Q; \Sigma)$  be an algebra. Consider  $Q[L] = \{v : Q \rightarrow L \mid v(a) \cdot v(b) = 0; a \neq b, \sum_{a \in Q} v(a) = 1\}$ . For every operation  $X$  of  $\Sigma$  define on  $Q[L]$  the following binary operation, which we denote by  $X_L$ :

$$X_L(\mu, \nu)(a) = \sum_{a=X(b,c)} \mu(b) \cdot \nu(c).$$

Denote  $\Sigma_L = \{X_L \mid X \in \Sigma\}$ .

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**Definition 4.** The algebra  $S[L] = (Q[L]; \Sigma_L)$  is called  $L$ -power of  $S$  or quasi-boolean power of  $S$ .

**Definition 5.** If  $L$  is a complete boolean algebra, then the algebra  $S[L]$  is called boolean power of  $S$ .

**Definition 6.** Let  $(A; \Sigma)$  and  $(B; \Theta)$  be algebras,  $\varphi : A \rightarrow B$  and  $\tilde{\psi} : \Sigma \rightarrow \Theta$  be mappings such that  $X$  and  $\tilde{\psi}(X)$  have the same arity. The pair  $(\varphi, \tilde{\psi})$  is called bihomomorphism from algebra  $(A; \Sigma)$  to algebra  $(B; \Theta)$ , if the condition

$$\varphi(X(a_1, \dots, a_n)) = \tilde{\psi}(X)(\varphi(a_1), \dots, \varphi(a_n))$$

holds for any  $X \in \Sigma$  and for any  $a_1, \dots, a_n \in A$  [2].

**Definition 7.** Let  $(A; \Sigma)$  and  $(B; \Theta)$  be algebras with binary operations and  $(\varphi, \tilde{\psi}) : (A; \Sigma) \rightarrow (B; \Theta)$  be a bihomomorphism. The bihomomorphism  $(\varphi, \tilde{\psi})$  is called commutative bihomomorphism, if the following condition

$$\tilde{\psi}(X)(\varphi(a), \varphi(b)) = \tilde{\psi}(X)(\varphi(b), \varphi(a))$$

holds for any  $X \in \Sigma$  and for any  $a, b \in A$ .

For the second order formulae (and the second order languages) see [3–5]. Let us recall, that a hyperidentity [2, 6–10] (or  $\forall(\forall)$ -identity) is a second-order formula of the following form:

$$\forall X_1, \dots, X_m \forall x_1, \dots, x_n (\omega_1 = \omega_2), \quad (*)$$

where  $\omega_1, \omega_2$  are words (terms) in the alphabet of functional variables  $X_1, \dots, X_m$  and objective variables  $x_1, \dots, x_n$ . However hyperidentities are usually presented without universal quantifiers:  $\omega_1 = \omega_2$ . The hyperidentity  $\omega_1 = \omega_2$  is said to be satisfied in the algebra  $(Q; \Sigma)$ , if this equality holds whenever every object variable  $x_j$  is replaced by an arbitrary element from  $Q$  and every functional variable  $X_i$  is replaced by an arbitrary operation of the corresponding arity from  $\Sigma$ . The possibility of such replacement is supposed, that is

$$\{|X_1|, \dots, |X_m|\} \subseteq \{|A| \mid A \in \Sigma\} = T_{(Q; \Sigma)} = T_{(\Sigma)},$$

where  $|S|$  is the arity of  $S$ , and  $T_{(Q; \Sigma)}$  is called the arithmetic type of  $(Q; \Sigma)$ . A  $T$ -algebra is an algebra with arithmetic type  $T \subseteq N$ . A class of algebras is called a class of  $T$ -algebras, if every algebra in it is a  $T$ -algebra.

The hyperidentity is said to be non-trivial if  $m > 1$ , and it is trivial if  $m = 1$ . The number  $m$  is called the functional rank of the given hyperidentity (coidentity).

A binary algebra  $(Q; \Sigma)$  is said to be a  $q$ -algebra ( $e$ -algebra), if there is an operation  $A \in \Sigma$  such that  $Q(A)$  is a quasigroup (a groupoid with a unit). A binary algebra  $(Q; \Sigma)$  is called non-trivial if  $|\Sigma| > 1$ . It is known [2, 6] (see also [7, 11]), that if an associative non-trivial hyperidentity is satisfied in a non-trivial  $q$ -algebra ( $e$ -algebra), then this hyperidentity can only be of the functional rank 2 and of one of the following forms:

$$X(x, Y(y, z)) = Y(X(x, y), z), \quad (ass)_1$$

$$X(x, Y(y, z)) = X(Y(x, y), z), \quad (ass)_2$$

$$Y(x, Y(y, z)) = X(X(x, y), z). \quad (ass)_3$$

Moreover, in the class of  $q$ -algebras ( $e$ -algebras) the hyperidentity  $(ass)_3$  implies the hyperidentity  $(ass)_2$ , which, in turn, implies the hyperidentity  $(ass)_1$ .

A binary algebra  $(Q; \Sigma)$  is called hyperassociative, if it satisfies the following hyperidentity of associativity:

$$X(x, Y(y, z)) = Y(X(x, y), z). \quad (ass)_1$$

**Theorem 1.** Let  $S = (Q; \Sigma)$  be a hyperassociative algebra and  $L(+, \cdot)$  be a complete boolean algebra. Then  $S[L] = (Q[L]; \Sigma_L)$  is a hyperassociative algebra.

**Proof.** We have

$$Q[L] = \left\{ v : Q \rightarrow L \mid v(a) \cdot v(b) = 0; a \neq b, \sum_{a \in Q} v(a) = 1 \right\}.$$

We should proof the following hyperidentity of associativity:

$$X_L(\mu, Y_L(v, \tau)) = Y_L(X_L(\mu, v), \tau). \quad (1)$$

Take any  $a \in Q$ :

$$\begin{aligned} X_L(\mu, Y_L(v, \tau))(a) &= \sum_{a=X(b,c)} \mu(b) \cdot Y_L(v, \tau)(c) = \\ &= \sum_{a=X(b,c)} \mu(b) \cdot \left( \sum_{c=Y(d,e)} v(d) \cdot \tau(e) \right) = \sum_{a=X(b,Y(d,e))} \mu(b) \cdot (v(d) \cdot \tau(e)), \end{aligned} \quad (2)$$

$$\begin{aligned} Y_L(X_L(\mu, v), \tau)(a) &= \sum_{a=Y(b,c)} X_L(\mu, v)(b) \cdot \tau(c) = \\ &= \sum_{a=Y(b,c)} \left( \sum_{b=X(d,e)} \mu(d) \cdot v(e) \right) \cdot \tau(c) = \sum_{a=Y(X(d,e),c)} (\mu(d) \cdot v(e)) \cdot \tau(c). \end{aligned} \quad (3)$$

Note, that the last equalities in (2) and (3) follows from the distributivity of  $L$ . According to the associativity of  $\cdot$  in  $L$ , hyperassociativity of  $S$  and (2), (3), we get (1).  $\square$

#### Auxiliary Results and Concepts.

**Theorem 2.** [10]. The complete, complemented lattice  $L(+, \cdot)$  will be quasi-boolean if and only if it admits  $(\cdot)$ -homomorphism on some complete boolean lattice. Therefore the homomorphism is one-to-one in the 0 and 1 and preserves l.u.b.'s of orthogonal systems. Such homomorphism is called canonical.

**Lemma 1.** Let  $S = (Q; \Sigma)$  be a rectangular algebra [12] and  $L_0(+, \cdot)$  is a complete boolean algebra. Then  $(S[L_0]; \Sigma_{L_0})$  is a rectangular algebra.

**Proof.** We should show that  $X_{L_0}(\mu, X_{L_0}(v, \mu)) = \mu$ . Indeed, take any  $a \in Q$ :

$$\begin{aligned} X_{L_0}(\mu, X_{L_0}(v, \mu))(a) &= \sum_{a=X(b,c)} \mu(b) \cdot X_{L_0}(v, \mu)(c) = \\ &= \sum_{a=X(b,c)} \mu(b) \cdot \left( \sum_{c=X(d,e)} v(d) \cdot \mu(e) \right) = \sum_{a=X(b,X(d,e))} (\mu(b) \cdot v(d) \cdot \mu(e)) = \\ &= \sum_{a=X(b,X(d,b))} (\mu(b) \cdot v(d) \cdot \mu(b)) = \sum_{d \in Q} \mu(a) \cdot v(d) = \end{aligned}$$



$$= \mu(a) \cdot \left( \sum_{d \in Q} \nu(d) \right) = \mu(a) \cdot 1 = \mu(a).$$

□

**Lemma 2.** Let  $Q$  be a relation defined on an idempotent hyperassociative algebra  $S = (A; \Sigma)$  such that for all  $a, b \in A$   $(X(a, b), X(b, a)) \in Q$ . Then  $\theta^* \subseteq Q$ , where  $\theta^* = \{(x, y) \in Q \times Q \mid X(x, X(y, x)) = x, X(y, X(x, y)) = y, \forall X \in \Sigma\}$ .

**Proof.** Let  $(a, b) \in \theta^*$ . Then

$$\begin{aligned} Y(X(a, b), X(b, a)) &= X(a, Y(b, X(b, a))) = \\ &= X(a, X(Y(b, b), a)) = X(a, X(b, a)) = a. \end{aligned}$$

In the same way we can get that  $Y(X(b, a), X(a, b)) = b$ .

Because  $(Y(X(a, b), X(b, a)), Y(X(b, a), X(a, b))) \in Q$ , so  $(a, b) \in Q$ . □

**Main Results.**

**Theorem 3.** Let  $B = (A; \Sigma)$  be an algebra,  $S = (Q; \Sigma)$  be a rectangular algebra and  $L(+, \cdot)$  be a quasi-boolean lattice. If  $B$  is embedded into  $(S[L], \Sigma_L)$ , then  $B$  will be idempotent and with transitive commutativity property [12].

**Proof.** Let us prove that  $(S[L], \Sigma_L)$  is an idempotent algebra. Indeed, take any  $v \in S[L]$ , any  $X_L \in \Sigma_L$  and any  $a \in Q$ :

$$\begin{aligned} X_L(v, v)(a) &= \sum_{a=X(b, c)} \nu(b) \cdot \nu(c) = \sum_{a=X(b, b)} \nu(b) \cdot \nu(b) = \\ &= \sum_{a=b} \nu(b) = \nu(a) \rightarrow X_L(v, v) = v. \end{aligned}$$

Now indicate, that  $(S[L], \Sigma_L)$  has a transitive commutativity property.

According to the Theorem [2], there exist complete boolean lattices  $L_0$  and  $f_0 : L \rightarrow L_0$  is a canonical  $(\cdot)$ -homomorphism from a quasi-boolean lattice  $L$  to a complete boolean lattice  $L_0$ .

Consider the following map  $f : S[L] \rightarrow S[L_0]$ :

$$f(v) = v \circ f_0.$$

Let us show that the map  $f$  is a homomorphism. We should prove that for  $\forall \mu, \nu \in S[L], \forall X \in \Sigma$

$$f(X_L(\mu, \nu)) = X_{L_0}(f(\mu), f(\nu)). \quad (4)$$

Indeed, take any  $a \in Q$

$$f(X_L(\mu, \nu))(a) = f_0(X_L(\mu, \nu)(a)) = f_0 \left( \sum_{a=X(b, c)} \mu(b) \cdot \nu(c) \right). \quad (5)$$

Consider the following system  $\lambda_a^X = \{\mu(b) \cdot \nu(c) \mid X(b, c) = a\}$ . According to the commutativity of operation  $\cdot$  in lattice  $L$ , we have that  $\lambda_a^X$  is an ortogonal system. So, because the map  $f_0$  saves the supremums of ortogonal systems, we have

$$f_0 \left( \sum_{a=X(b, c)} \mu(b) \cdot \nu(c) \right) = \sum_{a=X(b, c)} f_0(\mu(b)) \cdot f_0(\nu(c)).$$

So, according to the equalities (5), we have

$$f(X_L(\mu, \nu))(a) = \sum_{a=X(b,c)} f_0(\mu(b)) \cdot f_0(\nu(c)). \quad (6)$$

On the other hand,

$$X_{L_0}(f(\mu), f(\nu))(a) = \sum_{a=X(b,c)} f(\mu)(b) \cdot f(\nu)(c) = \sum_{a=X(b,c)} f_0(\mu(b)) \cdot f_0(\nu(c)). \quad (7)$$

According to equalities (6) and (7), we get (4).

Let  $\theta = \{(\mu, \nu) \in S[L] \mid X_L(\mu, \nu) = X_L(\nu, \mu); \forall X_L \in \Sigma_L\}$  is a relation of commutativity in  $S[L]$ . Let us show that  $\ker(f) = \theta$ .

We have  $\ker(f) = \{(\mu, \nu) \in S[L] \mid f(\mu) = f(\nu)\}$ . It is enough to show that  $f(\mu) = f(\nu) \Leftrightarrow X_L(\mu, \nu) = X_L(\nu, \mu); \forall X_L \in \Sigma_L$ .

Let  $f(\mu) = f(\nu)$ . We show that  $X_L(\mu, \nu) = X_L(\nu, \mu), \forall X_L \in \Sigma_L$ .

Indeed, consider the product  $\mu(a) \cdot \nu(b)$ , where  $a \neq b$ . We have

$$\begin{aligned} f_0(\mu(a) \cdot \nu(b)) &= f_0(\mu(a)) \cdot f_0(\nu(b)) = f_0(\nu(a)) \cdot f_0(\nu(b)) = \\ &= f_0(\nu(a) \cdot \nu(b)) = f_0(0) = 0. \end{aligned}$$

So  $f_0(\mu(a) \cdot \nu(b)) = 0$ . Since  $f_0$  is one-to-one in 0, we get  $\mu(a) \cdot \nu(b) = 0$ .

Take any  $X_L \in \Sigma_L$  and any  $a \in Q$ :

$$\begin{aligned} X_L(\mu, \nu)(a) &= \sum_{a=X(b,c)} \mu(b) \cdot \nu(c) = \sum_{a=X(b,b)} \mu(b) \cdot \nu(b) = \mu(a) \cdot \nu(a) = \nu(a) \cdot \mu(a) = \\ &= \sum_{a=X(b,c)} \nu(b) \cdot \mu(c) = X_L(\nu, \mu)(a) \Rightarrow X_L(\mu, \nu) = X_L(\nu, \mu). \end{aligned}$$

Conversely, let  $X_L(\mu, \nu) = X_L(\nu, \mu); \forall X_L \in \Sigma_L$ . We have

$$X_{L_0}(f(\mu), f(\nu)) = f(X_L(\mu, \nu)) = f(X_L(\nu, \mu)) = X_{L_0}(f(\nu), f(\mu)). \quad (8)$$

Since the rectangular algebra is anticommutative, according to the Lemma 1 and the equality (8), we get that  $f(\mu) = f(\nu)$ . So  $\ker(f) = \theta$ .

The  $\ker(f)$  is a congruence on  $(S[L], \Sigma_L)$ . So  $\theta$  is transitive.

We get that  $(S[L], \Sigma_L)$  is idempotent and has a transitive commutativity property. So each embedded algebra is idempotent and has transitive commutativity property.  $\square$

**Theorem 4.** Let  $S = (A; \Sigma)$  be an idempotent and hyperassociative algebra. Then there exist an idempotent, commutative algebra  $T = (B; \Theta)$  and a weak bihomomorphism  $(\varphi, \tilde{\psi}) : S \rightarrow T$  such that the inverse image of any element of  $T$  is an anticommutative idempotent semigroup. The bihomomorphism  $(\varphi, \tilde{\psi})$  is the weak in the sense that for any other commutative bihomomorphism  $(\Phi, \tilde{\Xi})$  from  $S$  to  $T$  the following condition holds:

$$\ker \varphi \subseteq \ker \Phi.$$

**Proof.** Consider the following relation on  $S$ :

$$\theta^* = \left\{ (x, y) \in Q \times Q \mid X(x, X(y, x)) = x, X(y, X(x, y)) = y, \forall X \in \Sigma \right\}.$$

$$\theta_a^* = \{x \in Q \mid (a, x) \in \theta^*\},$$

$$A/\theta^* = \{\theta_a^* \mid a \in A\},$$

$$\Sigma/\theta^* = \left\{ X^* : A/\theta^* \times A/\theta^* \rightarrow A/\theta^* \mid X^*(\theta_a^*, \theta_b^*) = \theta_{X(a,b)}^*, X \in \Sigma \right\}.$$

It is shown that  $\theta^*$  is a congruence and the quotient algebra  $S^* = (A/\theta^*, \Sigma/\theta^*)$  is an idempotent, commutative algebra and its elements are rectangular semigroups, so also anticommutative, idempotent semigroups (see [12]).

Consider the mapping  $(\varphi, \tilde{\psi}) : S \rightarrow S^*$ :

$$\varphi : A \rightarrow A/\theta^*, \varphi(a) = \theta_a^*,$$

$$\tilde{\psi} : \Sigma \rightarrow \Sigma/\theta^*, \tilde{\psi}(X) = X^*,$$

which is evidently bihomomorphism. Let us show that  $\varphi^{-1}(\theta_a^*)$  is an anticommutative idempotent semigroup for every  $\theta_a^* \in A/\theta^*$ . Indeed,

$$\varphi^{-1}(\theta_a^*) = \left\{ b \in A \mid (a, b) \in \theta^* \right\} = \theta_a^*,$$

so since  $\theta_a^*$  is an idempotent anticommutative semigroup,  $\varphi^{-1}(\theta_a^*)$  is also an idempotent anticommutative semigroup.

Let  $(\Phi, \tilde{\Xi})$  be a commutative bihomomorphism of  $S$ . Let us show that  $\ker \varphi \subseteq \ker \Phi$ . Indeed, it is easy to see that  $\ker \varphi = \theta^*$  and  $\ker \Phi$  is a congruence. It is evident also that  $(X(a, b), X(b, a)) \in \ker \Phi$ , because the bihomomorphism  $(\Phi, \tilde{\Xi})$  is commutative. It remains to use Lemma 2.  $\square$

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ՆԱՆՐԱՏԱԾԻՎՆԵՐԻ ՔՎԱԶԻԲՈՒՅԱՆ ԱՍՏԻՃԱՆ ԵՎ  
ԻՆՔՆԱՏԱՍՏՆԿՆՈՂ ՆԱՆՐԱՏԱԾԻՎՆԵՐ

Աշխատանքում ապացուցվում է հանրահաշվի ուղղանկյուն հանրահաշվի քվազիբոյան աստիճյանի մեջ ներդրվելու համար անհրաժեշտ պայման: Ապացուցվում է նաև, որ ցանկացած ինքնահամընկնող գերզուգորդական հանրահաշիվ թույլ բիհոմոնորֆիզմի միջոցով արտապարկվում է ինքնահամընկնող քվեդափոխական հանրահաշվի մեջ: